COMBINATORIAL RESULTS IN FLUCTUATION THEORY*

BY

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1. INTRODUCTION

Takacs [5,6] considered the sequence $v_1, v_2, \ldots, v_n$ of interchangeable random variables assuming non-negative integral values and derived the distributions of the statistics

$$\Delta_n,\ \Delta_n^*,\ \Delta_n^{(c)} \text{ and } \Delta_n^{(-c)}$$

concerning the partial sums $N_r = v_1 + \ldots + v_r$, $r = 1, 2, \ldots, n$.

In the present paper we shall derive, for $c > 0$, the distributions of the following statistics *viz.*,

$\Lambda_n^{(c)}$: number of subscripts $r = 1, 2, \ldots, n$ for which $N_r = r + c$ holds,

$\rho_n^{(c)}$: number of subscripts $r = 1, 2, \ldots, n$ for which $N_{r-1} = N_r = r + c$ holds,

$\nabla_n^{(c)}$: number of subscripts $r = 1, 2, \ldots, n$ for which $N_{r-1} \leqslant r + c - 1$, $N_r = r + c$ holds,

$\nabla_n^{*(c)}$: number of subscripts $r = 1, 2, \ldots, n$ for which $N_{r-1} = r + c$, $N_r > r + c$ holds,

under the condition that $N_n = v_1 + \ldots + v_n = k$ is fixed, where $v_1, v_2, \ldots, v_n$ is a sequence of mutually independent and identically distributed random variables having 'Geometric-Distribution', *i.e.*,

$$P\{v_r = i\} = pq^i,\ i = 0, 1, 2, \ldots \qquad \ldots(1{\cdot}1)$$

where $\qquad p + q = 1,\ 0 < p < 1.$

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Further, we define

$$\Lambda_n^{(-c)}\ ,\ \rho_n^{(-c)}\ ,\ \nabla_n^{(-c)},\ \text{and}\ \nabla_n^{*(-c)}$$

analogous to

$$\Lambda_n^{(c)}\ ,\ \rho_n^{(c)}\ ,\ \nabla_n^{(c)}\ \text{and}\ \nabla_n^{*(c)}$$

on replacing c by $-c$ in the definition. In the sequel we shall employ the technique of path methods as suggested by Csaki and Vincze [1] and Kanwar Sen [3, 4]. Applications of the results have been mentioned in section 5.

2. A Lattice Path Representation

Set $N_r = v_1 + \ldots + v_r$ for $r = 1, \ldots, n$ and $N_0 = 0$.

Then

$$P\ \{N_n = j\} = \binom{n+j-1}{n-1} p^n q^j \text{ for } j = 0, 1, \ldots \qquad \ldots(2\cdot1)$$

Let us represent the sequence $v_1, v_2, \ldots, v_n$ of non-negative integers by a minimal lattice path in the following manner : (*i*) the path starts from the origin ; (*ii*) for every j, v_j represents one horizontal unit followed by v_j vertical units and the section of the path contributed by v_j starts where the section of the path contributed by v_{j-1} ended (*see Fig. 1a*). We call such a path, from $(0, 0)$ to (n, N_n), a minimal lattice path with vartices (r, N_r), $r = 1, \ldots, n$.

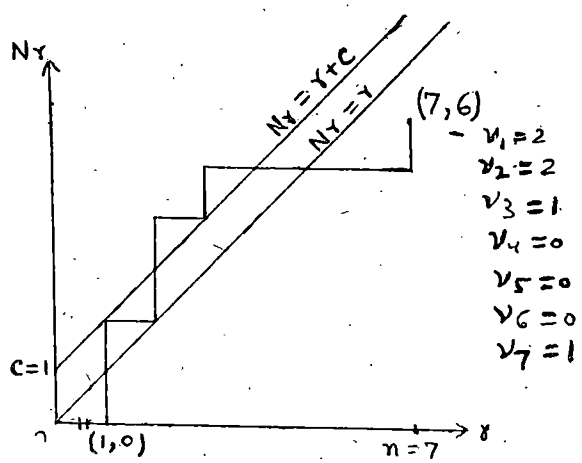


Fig. 1*a*

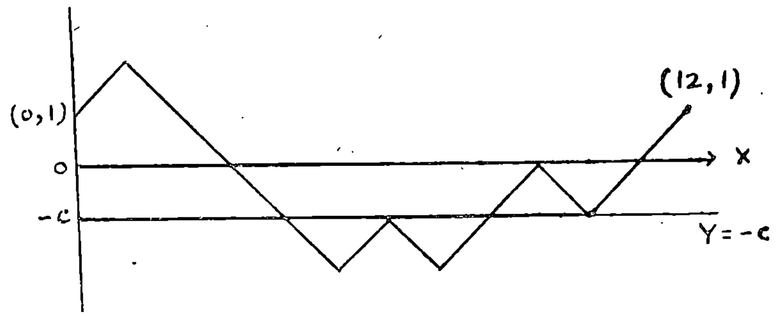


Fig. 1*b*.

We observe from (1·1) and (2·1) that the sequence $v_1, \ldots, v_n$ of geometric random variables possesses the property that all possible lattice paths $(N_1,\ldots,N_n)$ from (0, 0) to (n, N_n) are equally likely each with probability $P(N_n=k)\Big/\binom{n+k-1}{n-1}=p^n q^k$.

On rotating the lattice path (*Fig. 1a*), from (0, 0) to (n, N_n), obtained by the above construction, through 225° about the origin (so that the starting point becomes the end point and *vice versa*) and referring the line $N_r=r$ as x-axis, we observe that it is equivalent to a simple random walk path, as defined below in section 3, from $(0, n-N_n)$ to $(n+N_n-1, 1)$ (*see Figs 1a and 1b*). This procedure of rotation will henceforth be referred to as the "rotation procedure".

3. NOTATIONS

Let $\theta_1, \theta_2, \ldots, \theta_n$ be independent random variables with $P(\theta_i=+1)=q$, $P(\theta_i=-1)=p=1-q$, *i.e.* the sequence θ_i generates a simple random walk. This can be represented in an (i, S) coordinate system by a path with points (i, S_i), $i=0, 1, \ldots, n$ where $S_o=0$, $S_i=\theta_i+\ldots+\theta_i$ where the consecutive points are connected by straight lines.

For ease in writing we introduce the following symbols :

$R^{(t)}$ point : a point (i, S_i) where a path reaches the line $y=t$ *i.e.* for which $S_i=t$.

$R_+^{(t)}\left(R_-^{(t)}\right)$: an $R^{(t)}$ point (i, S_i) such that $S_{i-1}=t+1$ $(S_{i-1}=t-1)$.

$W^{(t)}=t$-wave : the segment of a path included between two consecutive $R^{(t)}$ points is called a t-wave.

$W_+^{(t)}\left(W_-^{(t)}\right)$: a t-wave with $S_j>t$ $(S_i<t)$ at the intervening positions.

$T^{(t)}$ point : a point (i, S_i) of the path for which $(S_i=t, S_{i-1}. S_{i+1}=t^2-1)$ holds. This will be called the intersection point in the line $y=t$ $(-\infty<t<\infty)$; $T^{(o)}=T$.

$T_+^{(t)}\left(T_-^{(t)}\right)$: a point (i, S_i) of the path for which $S_{i-1}=t+1$, $S_i=t$, $S_{i+1}=t-1$ $(S_{i-1}=t-1;\ S_i=t,\ S_{i+1}=t+1)$.

$S^{(t)}=t$-section : the segment of a path included between two consecutive $r^{(t)}$ points. An $S^{(t)}$ may consist of one or more $W^{(t)}$ of the same type.

$S_+^{(t)}\left(S_-^{(t)}\right)$: *a* t-section with $S_i\geqslant t$ $(S_i\leqslant t)$ in between.

$R_f^{(t)}$ $=t$-reflection : *a* t-reflection occurs at an index j when $S_j=t$, $S_{j-1}. S_{j+1}=(t+1)^2$ or $(t-1)^2$.

$R_{f+}^{(t)}\left(R_{f-}^{(t)}\right)$: a point (j, S_j) of the path for which $S_{j-1}=S_{j+1}=t+1=S_j+1$ $(S_{j-1}=S_{j+1}=t-1=S_j-1)$.

$K^{E_{m,n}}$: a path from (o, k) to (m, n); $_oE_{m,n}=E_{m,n}$

$KE^{r}_{m,\ n,\ (t)}$: an $KE_{m,n}$ path having r $R^{(t)}$ points.

$KE^{\cdot,\ p}_{m,\ n,\ (t)}$: an $KE_{m,n}$ path having p $T^{(t)}$ points.

$KE^{\cdot,\ \cdot,\ q}_{m,\ n,\ (t)}$: an $KE_{m,n}$ path having q $R_f^{(t)}$ points.

$KE^{r,\ p,\ q}_{m,\ n,\ (t)}$: an $KE^{r}_{m,\ n,\ (t)}$ path having p $T^{(t)}$ and q $R_f^{(t)}$—points.

Similarly

$$KE^{r+}_{m,\ n,\ (t)},\ KE^{\cdot,\ p+}_{m,\ n,\ (t)} \text{ and } KE^{\cdot,\ \cdot,\ q+}_{m,\ n,\ (t)}$$

denote an $KE_{m,n}$ path having exactly

$$\gamma\, R_+^{(t)},\ p\, T_+^{(t)} \text{ and } q\, R_{f+}^{(t)}$$

points respectively

$H_{m,n}$: an $E_{m,n}$ path reaching the height n for the first time at the mth step.

$N[A]$: the number of all A paths, *e.g.*

$$N[E_{m,n}]=\binom{m}{\frac{1}{2}(m-n)}$$

We shall use in the sequel the α, β, γ, δ and ε-operations involving translation or rotation or reflection of segments of paths as discussed in detail by Kanwar Sen [4].

4. Distribution of $\Lambda_n^{(c)}$, $\rho_n^{(c)}$, $\nabla_n^{(c)}$, and $\nabla_n^{*(c)}$ for a positive integer c

Theorem 1

$$\binom{2n+c-l-1}{n-1} P\left\{ \Lambda_n^{(c)} = j / N_n = n+c-l \right\}$$

$$= \frac{2j+c+l}{2n+c-l} \binom{2n+c-l}{n+c+j}, \quad l=0, 1, \ldots, n+c \quad \ldots(4{\cdot}1)$$

and

$$\binom{2n+c+l-1}{n-1} P\left\{ \Lambda_n^{(c)} = j / N_n = n+c+l \right\}$$

$$= \frac{2j+c+l+2}{2n+c+l} \binom{2n+c+l}{n-j-1}, \quad l=1, 2, \cdots(4{\cdot}2)$$

Proof

To prove (4·1), let $OPP_0P_1P_2\ldots P_j Q$ (see Fig. 2*a*) be a lattice path from 0 (0, 0) to $Q\,(n,\ n+c-l)$ with $Nr=r+c$ for exactly j indices, P_i $(i=1, \ldots, j)$ being the point (r, N_r) where $N_r=r+c$ for the ith time and $P=(1, 0)$. P_0 be the point where the lattice path meets the line $N_r=r+c$ for the first time.

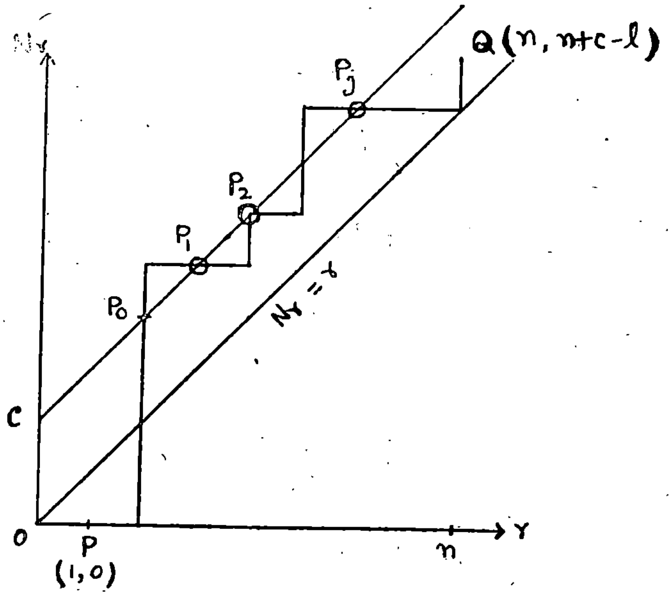
On applying the 'rotation procedure' we observe that (see Fig. 2*b*)

$$N\left[\Lambda_n^{(c)} = j,\ N_n = n+c-l \right] = N\left[-(c-l)\ E^{j+}_{2n+c-l-1,\ 1,\ (-c)} \right]$$

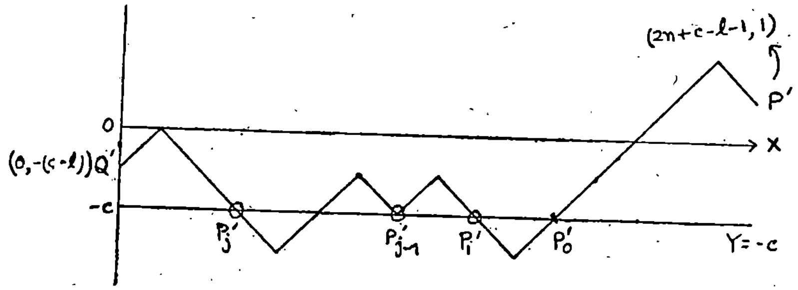
Now we establish below a 1 : 1 correspondence between

$$-(c-l)\ E^{j+}_{2n+c-l-1,\ 1,\ (-c)} \text{ paths and } H_{2n+c-l},\ 2j+c+l$$

paths. In the path $Q'P'_j\ldots P'_1P'_0P'$ (Fig. 2*b*) let us apply β-operation [4] to the segment $Q'P'_j$, δ operation [4] to $P'_jP'_1$ and γ-operation [4] to P'_1P' after reflecting the portion $P'_1P'_0$ (if any) in the line $y=-c$ and attach the transformed segments end-to-end in order. Finally let us now insert at the end (*i. e.* after $\theta_{2n+c-l-1}$) a $(+1)$. Thus we obtain an $H_{2n+c-l,\ 2j+c+l}$ path. By reversing the procedure it may be seen that this transformation is 1 : 1.



(Fig. 2*a*)



(Fig. 2*b*)

Hence

$$N\left[>^{(c)}_{n} = j,\ \ N_n = n+c-l \right] = N[H_{2n+c-l,\ 2j+c+l}]$$

leading to (4.1) for $l=0, 1, \ldots, (n+c)$ by Feller [2p.89].

Similarly by the rotation procedure we have for $l=1, 2, \ldots$

$$N\left[\wedge^{(c)}_{n} = j,\ N_n = n+c+l \right] = N\left[-(c+l)E^{j+}_{2n+c+l-1,\ 1,\ (-c)} \right]$$

$$= N\left[H_{2n+c+l,\ 2j+c+l+2} \right]$$

leading to (4.2) by [2].

Theorem 2

$$\binom{2n+c-l-1}{n-1} P\left\{\rho_n^{(c)}=j/N_n=n+c-l\right\}$$
$$=\left(\frac{4j+c+l+1}{2n+c-l+1}\right)\binom{2n+c+l+1}{n-2j-l},\ l=1,2,\ \ldots n+c \quad \ldots(4.3)$$

and $$\binom{2n+c+l-1}{n-1} P\left\{\rho_n^{(c)}=j/N_n=n+c+l\right\}$$
$$=\left(\frac{4j+c+l+3}{2n+c+l+1}\right)\binom{2n+c+l+1}{n-2j-1},\ l=0,1,2,\ \ldots(4.4)$$

Proof

To prove (4.3) let $OPP_0P_1\ldots P_j\ Q$ (see *Fig. 3a*) be the lattice path from $O(0, 0)$ to $Q(n, n+c-l)$ with $N_{r-1}=N_r=r+c$ for exactly j indices, $P_i\ (i=1, \ldots, j)$ being the lattice point (r, N_r) such that

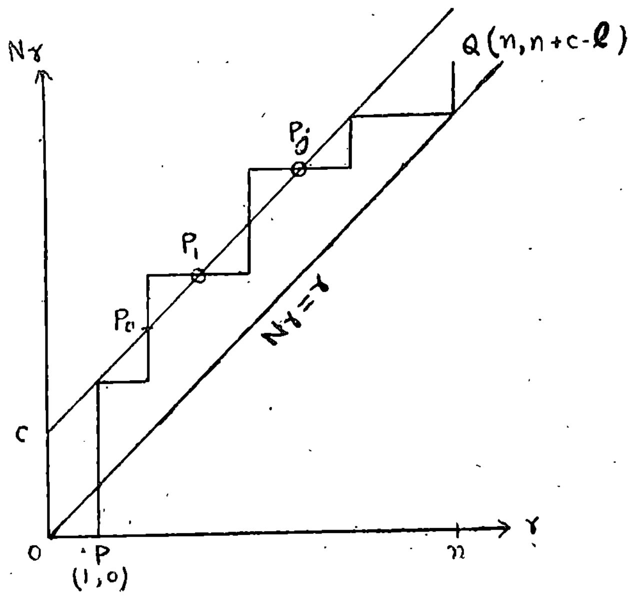


Fig. 3*a*

$N_{r-1}=N_r=r+c$ for the ith time. P_0 be the point where the lattice path rises the line $N_r=r+c$ for the first time.

On applying the 'rotation procedure' to the path *in Fig. 3a* we observe that the transformed path $Q'P'_j\ldots P'_1P'_0P'$ (see fig. 3*b*) is an

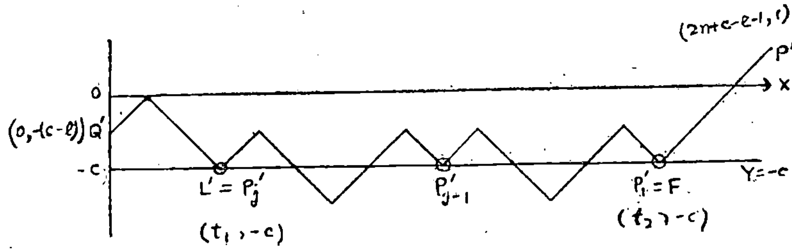


Fig. 3*b*

$-(c-l)E_{2n+c-l-1,\,1}$ path having exactly $j\ T_{+}^{(-c)}$ points and hence $2j\ T^{(-c)}$ points. Thus

$$N\left[\rho_n^{(c)}=j,\ N_n=n+c-l\right]=N\left[{}_{-(c-l)}\ E^{\cdot,2j}_{2n+c-l-1,\ 1,\ (-c)}\right]$$

Let $(t_1, -c)$ be the first and $(t_2, -c)$ the last $T^{(-c)}$ points (*see Fig. 3b*). Let us apply the β and γ-operations [4] to the segments $Q'P'_j$ and P'_oP' respectively. The δ-operation [4] on $P'_jP'_o$ yields the $H_{t_2-t_1,\ 4_j-2}$ path. Attach these transformed segments in order. Finally let us insert both between θ_{t_1} and θ_{t_1+1} and after $\theta_{2n+c-l-1}$ $a(+1)$. Thus we obtain a $H_{2n+c-l+1},\ 4j+c+l+1$ path. By reversing this procedure we see that the transformation is 1 : 1. Therefore,

$$N\left[\ \rho_n^{(c)}=j,\ N_2=n+c-l\ \right]=N\left[H_{2n+c-l+1},\ 4j+c+l+1\right)$$

leading to (4·3) by [2].

Similarly by the analogous arguments it can easily be shown that for l=0, 1, 2, ...

$$N\left[\ \rho_n^{(c)}=j,\ N_n=n+c+l\ \right]=N\left[-(c+l)\ E^{\cdot,\ j+}_{2n+c+l-1,\ 1,\ (-c)}\right]$$

$$=N\left[-(c+l)\ E^{\cdot,\ 2j}_{2n+c+l-1,\ 1,\ (-c)}\right]$$

$$=N\left[H_{2n+c+l+1},\ 4j+c+l+3\right]$$

which leads to (4·4) by [2].

Theorem 3

$$\binom{2n+c-l-1}{n-1} P\left\{\nabla_n^{(c)}=j/N_n=n+c-l\right\} = \sum_{m=0}^{[\frac{1}{2}(n-l-j)]} \frac{j+c+l+3m}{2n+c-l-j-m}\binom{j+m}{m}$$

$$\binom{2n+c-l-j-m}{n+c+m},\ l=1,\ 2,\ \ldots,\ (n+c) \qquad \ldots(4.5)$$

and

$$\binom{2n+c+l-1}{n-1} P\left\{\nabla_n^{(c)}=j/N_n=n+c+l\right\} = \sum_{m=0}^{[\frac{1}{2}(n-j-1)]} \frac{j+c+l+3m+2}{2n+c+l-j-m}\binom{j+m}{m}$$

$$\binom{2n+c+l-j-m}{n-2m-j-1},\ l=0,\ 1,\ 2,\ \ldots \qquad \ldots(4.6)$$

where $[Z]$ denotes the greatest integer contained in Z.

To prove (4.5) let $OPPP_1\ldots P_jLQ$ (*see Fig. 4a*) be the lattice path from (0, 0) to $(n,\ n+c-l)$ with $(N_{r-1}\leqslant r+c-1,\ N_r=r+c)$ for

Proof

exactly j indices, P_i $(i=1, ..., j)$ being the point (r, N_r) such that $(N_{r-1} \leqslant r+c-1, N_r=r+c)$ for the ith time. F and L be the points where the lattice path meets the line $N_r=r+c$ for the first and the last time respectively.

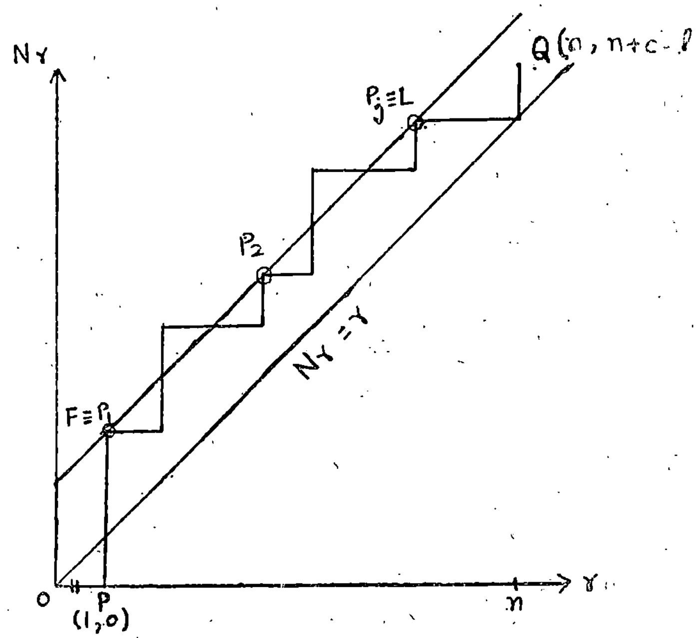


Fig. 4*a*

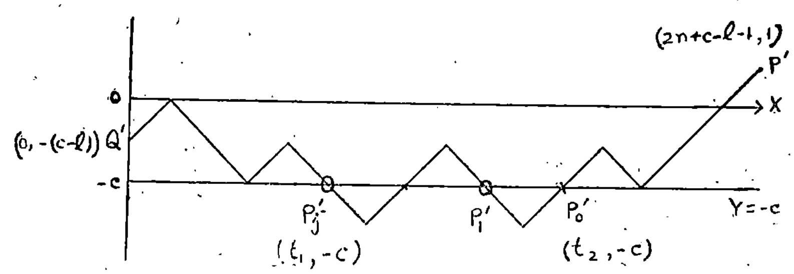


Fig. 4*b*

On applying 'the rotation procedure' we observe that (*see Fig. 4b*)

$$N\left[\nabla_n^{(c)}=j,\ N_n=n+c-l\right]=N\left[-(c-l)\,E^{\cdot,\ \cdot,\ j+}_{2n+c-l-1,\ 1,\ (-c)}\right]$$

$$=\sum_m N\left[-(c-l)\,E^{\cdot,\ m+,\ j+}_{2n+c-l-1,\ (-c)}\right]$$

$$=\sum_m N\left[-(c-l)\,E^{(m+j)+,\ 2m,\ j+}_{2n+c-l-1,\ 1,\ (-c)}\right] \qquad \ldots(4.7)$$

In Fig. 4*b*, L' $(t_1, -c)$ and F' $(t_2, -c)$ (where $t_2 > t_1$) be the points of the first and the last contact of the path with the line $y = -c$. The enumeration of paths on the right side of (4·7) includes the following four mutually exclusive cases : When

(*i*) L' is an $R_{f+}^{(-c)}$ (*i.e.* $L' = P'_j$) and F' is an $R_{f+}^{(-c)}$ (*i.e.* $F' = P'_1$)

(*ii*) L' is an $R_{f+}^{(-c)}$ and F' is a $T_{-}^{(-c)}$

(*iii*) L' is a $T_{+}^{(-c)}$ and F' is an $R_{f+}^{(-c)}$

(*iv*) L' is a $T_{-}^{(-c)}$ and F' is a $T_{-}^{(-c)}$.

Let D_1, D_2, D_3 and D_4 denote respectively the required number of paths from Q' to P' with j $R_{f+}^{(-c)}$ in cases (*i*), (*ii*), (*iii*) and (*iv*).

Then

$$N\left[\nabla_n^{(c)} = j, N_n = n + c - l\right] = \sum_{i=1}^{4} D_i \qquad \ldots(4.8)$$

We now consider our path $Q'L'P_j' \ldots P_1'F'P'$ as divided into three segments *viz.*, $Q'L'$, $L'F'$ and $F'P'$. For case (*i*) the corresponding path is shown in Fig. 4*b*. Here $L'F'$ contains $(j-2)$ $R_{f+}^{(-c)}$ and m $T_{+}^{(c)}$. Thus $L'F'$ consisting of $(2m+1)$ $S^{(-c)}$ [*i.e.* $(m+1) S_{+}^{(-c)}$ and m $S_{-}^{(-c)}$] includes $(j+m-1) W_{+}^{(c)}$ (of length $2p$, say). Since each $S_{+}^{(-c)}$ consists of at least one $W_{+}^{(-c)}$, $(m+1)$ $S_{+}^{(-c)}$ can be constructed out of $(j+m-1)$ ordered $W_{+}^{(-c)}$ in $\binom{j+m-2}{m}$ ways (from the occupancy problem of [2]).

Now the β, ε, δ and γ-operations [4] on $Q'L'$, $(j+m-1)$ $W_{+}^{(-c)}$, m $S_{-}^{(-c)}$ and $F'P'$ yield respectively $H_{t_1, l}$, $H_{2p-j-m+1, j+m-1}$, $H_{t_2-t_1-2p, 2m}$ and $H_{2n+c-l-1-t_2, c+1}$. On joining these transformed segments in order we get finally the $H_{2n+c-l-m-j, 3m+j+l+c}$ path where $0 \leqslant m \leqslant [\frac{1}{2}(n-l-j)]$. Thus

$$D_1 = \sum_{m=0}^{[\frac{1}{2}(n-l-j)]} \binom{j+m-2}{m} N[H_{2n+c-l-m-j}, 3m+j+l+c] \ldots(4.9)$$

Similarly we can show that

$$D_2 = D_3 = \sum_{m=1}^{[\frac{1}{2}(h-l-j)]} \binom{j+m-2}{m-1} N[H_{2n+c-l-m-j}, 3m+j+l+c] \quad \dots(4.10)$$

$$D_4 = \sum_{m=2}^{[\frac{1}{2}(n-l-j)]} \binom{j+m-2}{m-2} N[H_{2n+c-l-m-j}, 3_{m+j+l+c}] \quad \dots(4.11)$$

Hence (4.5) follows from (4.8) to (4.11) by using [2]. In a similar fashion the result (4.6) can also be proved.

Now we state below another result. Its proof follows immediately from the argument to that given in the proceeding theorem.

Theorem 4

$$\binom{2n+c-l-1}{n-1} P\left\{\nabla_n^{*(c)} = j/N_n = n+c-l\right\} = \sum_{m=1}^{[\frac{1}{2}(n-l-j)]} \frac{j+c+l+3m+1}{2n+c-l-j+m+1}$$

$$\binom{j+m-1}{j} \cdot \binom{2n+c-i-j-m+1}{n+c+m+1}, \; l=0, 1, \dots, (n+c) \quad \dots(4.12)$$

and

$$\binom{2n+c+l-1}{n-1} P\left\{\triangle_n^{*(c)} = j/N_n = n+c+l\right\} = \sum_{m=1}^{[\frac{1}{2}(n-j+1)]} \frac{j+c+l+3m-1}{2n+c-l-j-m+1}$$

$$\binom{j+m-1}{j} \binom{2n+c+l-j-m+1}{n+c+l+m}, \; l=1, 2, \dots \quad \dots(4\cdot13)$$

Finally we quote below other results concerning the distributions of the above statistics with superscript $(-c)$, where c is a positive integer.

$$\binom{2n-c-l-1}{n-1)} P\left\{\Lambda_n^{(-c)} = j/N_n = n-c-l\right\} = \frac{2j+c+l-2}{2n-c-l} \binom{2n-c-l}{n+j-1},$$

$$l = 0, 1, \; , (n-c) \quad \dots(4\cdot14)$$

$$\binom{2n-c+l-1}{n-1} P\left\{\Lambda_n^{(-c)}=j/N_n=n-c+l\right\}$$

$$=\frac{2j+c+l}{2n-c+l}\binom{2n-c+l}{n-c-j},$$

$$l=1, 2, \ldots \qquad \ldots(4{\cdot}15)$$

$$\binom{2n-c-l-1}{n-1} P\left\{\rho_n^{(-c)}=j/N_n=n-c-l\right\}$$

$$=\frac{4j+c+l-3}{2n-c-l+1}\binom{2n-c-l+1}{n-2j-1},$$

$$l=1, \ldots, (n-c). \qquad \ldots(4{\cdot}16)$$

$$\binom{2n-c+l-1}{n-1} P\left\{\rho_n^{(-c)}=j/N_n=n-c+l\right\}$$

$$=\frac{4j+c+l-1}{2n-c+l+1}\binom{2n-c+l+1}{n+l+2j},$$

$$l=0, 1, 2, \ldots \qquad \ldots(4{\cdot}17)$$

$$\binom{2n-c-l-1}{n-1} P\left\{\nabla_n^{(-c)}=j/N_n=n-c-l\right\}$$

$$=\sum_{m=1}^{[\frac{1}{2}N_n-j+2)}\frac{j+c+l+3m-3}{2n-c-l+1-j+m}\binom{j+m-1}{j}$$

$$\binom{2n-c-l-j-m+1}{n+m-1}, \quad l=1, \ldots, (n-c) \qquad \ldots(4{\cdot}18)$$

$$\binom{2n-c+l-1}{n-1} P\left\{\nabla_n^{(-c)}=j/N_n=n-c+l\right\}$$

$$=\sum_{m=1}^{[\frac{1}{2}(n-c-j+1)]}\frac{j+c+l+3m-1}{2n-c+l-j-m+1}$$

$$\binom{j+m-1}{j}\binom{2n-c+l-j-m+1}{n+m+l},$$

$$l=0, 1, \ldots \qquad \ldots(4{\cdot}19)$$

$$\binom{2n-c-l-1}{n-1} P\left\{\nabla_n^{*(-c)}=j/N_n=n-c-l\right\}$$

$$=\sum_{m=0}^{[\frac{1}{2}(N_n-j)]}\frac{j+c+l+3m}{2n-c-l-j-m}\binom{j+m}{j}$$

$$\binom{2n-c-l-j-m}{n+m}, \quad l=0, 1, \ldots, (n-c) \qquad \ldots(4{\cdot}20)$$

$$\binom{2n-c+l-1}{n-1} P\left\{\nabla_n^{*(-c)} = j / N_n = n-c+l\right\} = \sum_{m=0}^{[\frac{1}{2}(n-c-j+1)]} \frac{j+c+l+3m-2}{2n-c+l-j-m} \binom{j+m}{j} \binom{2n-c-j-m+l}{n+m+l-1}, \quad l = 1, 2, \ldots \qquad \ldots(4{\cdot}21)$$

These results can similarly be proved as Theorem 1, 2, 3 and 4.

5. Application of The Results in Deriving The Ballot Problems

Putting $j=0$ and $n+c-l=k$ in (4·1) we get,

$$\binom{n+k-1}{n-1} P\left\{\Lambda_n^{(c)} = 0 / N_n = k\right\} = \frac{n-k+2c}{n+k} \binom{n+k}{n+c} \text{ for } k=0, 1, \ldots, n+c,$$

or

$$P\left\{N_r < r+c \text{ for } r=1, \ldots, n / N_n = k\right\} = \left(1 - \frac{k-c}{n+c}\right) \frac{\binom{n+k-1}{n+c-1}}{\binom{n+k-1}{n-1}}, \text{ for } k=0, 1, \ldots, n+c$$

For $c=0$, reduces to

$$P\{N_r < r \text{ for } r=1, \ldots\ldots, n / N_n = k\} = 1 - \frac{k}{n} \text{ for } k=0, 1, \ldots, n \quad (5.1)$$

thus verifying the Takacs' lemma ((7), p. 4; [5]) which is the generalization of the classical ballot theorem reformulated as below:

Suppose that in a ballot candidate A scores a votes and candidate B scores b votes and all the possible $\binom{a+b}{a}$ voting records are equally probable. Denote by α_r and β_r the number of votes registered for A and B respectively among the first r votes recorded. Let c and μ be non-negative integers. Define v_r, $r=1, 2, \ldots, a+b$, as follows :

$$v_r = \begin{cases} o & \text{if the } r^{th} \text{ vote is cast for } A. \\ \mu + r & \text{if the } r^{th} \text{ vote is cast for } B. \end{cases}$$

Now $v_1, v_2, \ldots, v_{a+b}$ are interchangeable random variables that assume non-negative integer values and

$$v_1 + v_2 + \ldots v_{a+b} = b(\mu+1),$$

Set $N_r = v_1 + \ldots + v_r$, $r = 1, 2, \ldots, a+b$. Since $N_r = (\mu+1)\beta_r$ and $r = \alpha_r + \beta_r$, the inequality $\alpha_r > \mu\beta_r$ holds if and only if $N_r < r$.

Now on putting $n = a+b$ and $k = b(\mu+1)$, (5.1) gives

$$P\{\alpha_r > \mu\beta_r \text{ for } r = 1, \ldots, a+b / N_{a+b} = b(\mu+1)\} = \frac{a-\mu b}{a+b} \text{ for } a \geqslant \mu_b, \text{ thus}$$

verifying the classical ballot theorem ((1), p. 2; [5]).

Similarly other results may be applied in deriving the ballot problems. The statistics

$$\Lambda_n^{(c)}, \quad \rho_n^{(c)}, \quad \nabla_n^{(c)}, \quad \text{and } \nabla_n^{*(c)}$$

are equivalent to certain characteristics of the ballot problem as given below :

$\Lambda_n^{(c)}$: number of subscripts $r = 1, \ldots, a+b$ for which $\alpha_r = \mu\beta_r - c$.

$\rho_n^{(c)}$: number of subscripts $r = 1, \ldots, a+b$ for which $\alpha_r = \mu\beta_r - c$ but $\alpha_{r-1} = \mu\beta_{r-1} - c - 1$.

$\nabla_n^{(c)}$: number of subscripts $r = 1, \ldots, a+b$ for which $\alpha_r = \mu\beta_r - c$: but $\alpha_{r-1} \geqslant \mu\beta_{r-1} - c$.

$\nabla_n^{*(c)}$: number of subscripts $r = 1, \ldots, a+b$ for which $\alpha_r < \mu\beta_r - c$ but $\alpha_{r-1} = \mu\beta_{r-1} - c - 1$.

SUMMARY

For non-negative integral valued interchangeable random variables, Takacs [5, 6] derived the distributions of statistics concerning their partial sums *viz.*,

$$\Delta_n, \ \Delta_n^*, \ \Delta_n^{(c)} \text{ and } \Delta_n^{(-c)}.$$

In this paper, we derive the distributions of some other statstics *viz.*,

$$\Lambda_n^{(c)}, \quad \Lambda_n^{(-c)}, \quad \rho_n^{(c)}, \quad \rho_n^{(-c)}, \quad \nabla_n^{(c)},$$

$$\nabla_n^{(-c)}, \quad \nabla_n^{*(c)} \text{ and } \nabla_n^{*(-c)} \quad (c > o)$$

for geometrically distributed independent random variables through the technique of path methods as discussed by Csaki and Vincze [1] and Kanwar Sen [3, 4].